Indian Statistical Institute, Bangalore

B. Math (Hons.) Second Year

Second Semester - Ordinary Differential Equations

Mid-Semester Exam | Date: February 19, 2025

Maximum Marks: 30 | Duration: 2 hours

Answer all questions

(1) Let φ, ψ, χ be real-valued continuous functions on a interval $I : a \le t \le b$. Let $\chi(t) > 0$ for all $t \in I$, and suppose for $t \in I$ that

$$\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s)ds.$$

Prove that

$$\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s)\exp\left(\int_s^t \chi(u)du\right)ds$$

for $t \in I$. [5]

(2) (a) Consider the equation $y' + ay = b(x)$, where a is a constant and $b(x)$ is a continuous function on an interval I. If x_0 is apoint in I and c is any constant, the function defined by

$$\varphi(x) = e^{-ax}\int_{x_0}^x e^{at}b(t)dt + ce^{-ax}$$

is a solution of this equation and every solutions are of this form. [2]

(b) Solve the system of equations $\frac{dx}{dt} = -4x - y$ and $\frac{dy}{dt} = x - 2y$. [3]

(3) (a) Let y_1 and y_2 be any two solutions of $y'' + P(x)y' + Q(x)y = 0$ and W be the Wronskian. Then show that $W = ce^{-\int P(x)dx}$ for some constant c. [3]

(b) Suppose that φ_1 and φ_2 are linearly independent solutions of the second order linear differential equation with constant coefficients

$$y'' + a_1y' + a_2y = 0$$

and W be the Wronskian. Show that W is constant if and only if $a_1 = 0$. [2]

(4) (a) Consider the IVP $y' = 2sin(3xy)$, $y(0) = y_0$. Show that it has unique solution in $(-\infty, \infty)$. [3]

(b) Find the solution of the IVP

$$y' = 2x(1 + y), \quad y(0) = 0$$

using Picard's iterations (successive approximations) $y_0(x), y_1(x), y_2(x), \ldots$ and show that it is unique around a region containing $(0, 0)$. [2]

(5) Find a function v such that y_1 and $y_2 = vy_1$ are linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0.$$

[5]

(6) (a) Define regular singular point of the equation $y'' + P(x)y' + Q(x)y = 0$. [1]

(b) Define the Bessel function $J_p(x)$ of first kind of order $p\,(\geq 0)$.

Show that

(i) $\frac{d}{dx}(x^p J_p(x)) = x^p J_{p-1}(x)$.

(ii) $J_{-p}(x) = (-1)^p J_p(x)$ for p positive integer.

[4]

Good luck!!